

# HALF-INVERSE SPECTRAL PROBLEMS FOR STURM-LIOUVILLE OPERATORS WITH SINGULAR POTENTIALS<sup>†</sup>

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**ABSTRACT.** Half-inverse spectral problem for a Sturm–Liouville operator consists in reconstruction of this operator by its spectrum and half of the potential. We give the necessary and sufficient conditions for solvability of the half-inverse spectral problem for the class of Sturm–Liouville operators with singular potentials from the space  $W_2^{-1}(0, 1)$  and provide the reconstruction algorithm.

## 1. INTRODUCTION

Assume that a function  $q$  is integrable and real-valued on  $(0, 1)$  and that  $h_0$  and  $h_1$  are some elements of  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ . Denote by  $T = T(q, h_0, h_1)$  a Sturm–Liouville operator in  $L_2(0, 1)$  that is given by the differential expression

$$(1.1) \quad \ell := -\frac{d^2}{dx^2} + q$$

and the boundary conditions

$$(1.2) \quad y'(0) - h_0 y(0) = 0, \quad y'(1) - h_1 y(1) = 0$$

(where, as usual,  $h_0 = \infty$  or  $h_1 = \infty$  is a shorthand notation for the Dirichlet boundary condition at the point  $x = 0$  or  $x = 1$  respectively). Suppose that the spectrum  $\Sigma(q, h_0, h_1)$  of  $T$ , the number  $h_0$  and/or  $h_1$ , and the potential  $q$  over a half of the interval—e.g., over  $(0, \frac{1}{2})$ —are known; can one recover the operator  $T$  based on this information? Problems of such kind are known in the literature as half-inverse spectral problems, or inverse problems with mixed spectral data.

The main aim of the present paper is to study the half-inverse spectral problem for Sturm–Liouville operators with real-valued singular potentials from the space  $W_2^{-1}(0, 1)$  (see Section 2 for precise definitions). Namely, we shall find necessary and sufficient conditions on mixed data in order that the half-inverse spectral problem be soluble in the considered class of potentials. We also specify this result to the case of regular potentials from  $L_2(0, 1)$  and establish then a local existence theorem.

The first result on the half-inverse spectral problem is due to Hochstadt and Lieberman [13], who proved that if  $\Sigma(\tilde{q}, \tilde{h}_0, \tilde{h}_1) = \Sigma(q, h_0, h_1)$ ,  $\tilde{h}_0 = h_0$ ,  $\tilde{h}_1 = h_1$ , and  $\tilde{q} = q$  on  $(0, \frac{1}{2})$ , then  $\tilde{q} = q$  on  $(0, 1)$ . Later, Hald [10] proved that the statement remains true even if the boundary conditions at the point  $x = 1$  are not assumed equal, while del Rio [5] constructed counterexamples demonstrating that uniqueness might fail if  $\tilde{h}_0 \neq h_0$ . Hald [11] generalized the theorem by Hochstadt and Lieberman to the setting motivated by the inverse problem for the torsional modes of the Earth, where the

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domain of  $T$  forces a discontinuity in an interior point. It was shown in [11] that not only the potential in  $(\frac{1}{2}, 1)$ , but also the position of the discontinuity and the jump magnitude are uniquely determined by the spectrum  $\Sigma(q, h_0, h_1)$ , the number  $h_0$ , and the potential  $q$  on  $(0, \frac{1}{2})$ . Willis [27] established similar uniqueness result for an inverse problem with two interior discontinuities, and Kobayashi [18] proved that a symmetric potential  $q \in L_1(0, 1)$  with two symmetrically placed jumps is uniquely determined from  $h_0$  and a single spectrum  $\Sigma(q, h_0, -h_0)$ . We also mention the paper [3], where the uniqueness for the half-inverse problem was established for the operator generated by the differential expression

$$\frac{1}{a(x)} \frac{d}{dx} \left( a(x) \frac{d}{dx} \right) + q(x)$$

with the impedance  $a$  vanishing at  $x = 0$  and positive otherwise.

Afterwards, these uniqueness results have been further generalized to different settings. In [6], del Rio, Gesztesy and Simon proved a number of results when  $q$  on a part of  $(0, 1)$  and certain parts of the spectra  $\Sigma(q, h_0, h_1)$  for several different values of  $h_1$  completely determine  $q$  on  $(0, 1)$ —e.g., so do the spectrum  $\Sigma(q, h_0, h_1)$ , half the spectrum  $\Sigma(q, h_0, h'_1)$  for  $h'_1 \neq h_1$ , and  $q$  on  $(0, \frac{1}{4})$ , or two-thirds of the spectra  $\Sigma(q, h_0, h_1)$  for three different values of  $h_1$ . In [7], Gesztesy and Simon proved that it suffices to know all the eigenvalues of  $T$  except  $k + 1$  provided the potential  $q$  is  $C^{2k}$ -smooth at the midpoint  $x = \frac{1}{2}$ . It was also proved in [7] that, under suitable growth conditions on the potential  $q$  of a Schrödinger operator  $T$  on the whole line, the spectrum of  $T$  and the potential on the half-line  $(0, \infty)$  determine uniquely  $q$  on the other half-line  $(-\infty, 0)$ . This last result was recently improved by Khodakovsky [19].

In the recent work [23], Sakhnovich studied existence of solution to the half-inverse spectral problem; namely, he presented sufficient conditions on a function  $q_0$  on  $(0, \frac{1}{2})$  and a sequence  $(\lambda_n^2)$  of pairwise distinct real numbers tending to  $+\infty$  in order that there existed a Sturm–Liouville operator  $T$  on  $(0, 1)$  with given spectrum  $\{\lambda_n^2\}$  and potential  $q$  coinciding with  $q_0$  on  $(0, \frac{1}{2})$ . These sufficient conditions are of local nature as they require that  $q_0$  belong to  $C^1[0, \frac{1}{2}]$  and that some quantities constructed through the mixed spectral data  $\{(\lambda_n^2), q_0\}$  be small enough; basically,  $q_0$  is required to have small enough norm in  $C^1[0, \frac{1}{2}]$  and the sequence  $(\lambda_n^2)$  to be close enough to the “unperturbed” one,  $(\pi^2 n^2)$ . Under these smallness assumptions the author also suggested a constructive algorithm based on iterative solution of a Gelfand–Levitan–Marchenko type integral equation.

We also mention the book by Pöschel and Trubowitz [22], where among other inverse results the authors prove that the odd part of a potential  $q \in L_2(0, 1)$  and the Dirichlet spectrum  $\Sigma(q, \infty, \infty)$  determine uniquely the whole potential. Coleman and McLaughlin [4] generalized the approach by Pöschel and Trubowitz and derived an analogous result for Sturm–Liouville operators in impedance form

$$Su = \frac{1}{p(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right)$$

with  $p \in W_2^1(0, 1)$ . Here  $p$  is a positive *impedance*, and the operator  $S$ , subject to suitable boundary conditions, is selfadjoint in the weighted space  $L_2((0, 1); p dx)$ .

Sturm–Liouville operators in impedance form were earlier treated by Andersson [1, 2]. In [1] the cases  $p \in W_r^1(0, 1)$ ,  $r \in [1, \infty)$ , and  $p$  of bounded variation were considered and various direct and inverse spectral problems addressed; in particular, uniqueness and local existence results for the half-inverse spectral problem were derived. In [2], among other direct and inverse spectral problems, a global existence result for

the half-inverse spectral problem for the impedance Sturm–Liouville operator  $S$  with  $p \in W_2^1(0, 1)$  was claimed. Namely, Theorem 5.2 of [2] states that for any positive function  $p_0 \in W_2^1(0, \frac{1}{2})$  and any sequence  $(\lambda_n^2)$  of pairwise distinct real numbers tending to  $+\infty$  and obeying the necessary asymptotics there exists an impedance Sturm–Liouville operator  $S$  with impedance  $p \in W_2^1(0, 1)$  extending  $p_0$ , whose spectrum coincides with the set  $\{\lambda_n^2\}$ . Unfortunately, as the example of Section 4 demonstrates, Theorem 5.2 of [2] is erroneous.

We observe that for  $p \in W_2^1(0, 1)$  the impedance Sturm–Liouville operator  $S$  is unitarily equivalent to a standard Sturm–Liouville operator with singular potential  $q := (\sqrt{p})''/\sqrt{p} \in W_2^{-1}(0, 1)$ . The theory of Sturm–Liouville operators with singular potentials from  $W_2^{-1}(0, 1)$  have been thoroughly developed in the recent works by Shkalikov and Savchuk (see, e.g., [25, 26]), and some settings of inverse spectral problems for such operators have been completely solved in [15, 16, 17].

The present work was highly motivated by the papers [2, 23]: we wanted, firstly, to correct the erroneous statement of Theorem 5.2 in [2] concerning solvability of the half-inverse spectral problem and, secondly, to formulate local existence result analogous to that of [23] directly in terms of the mixed data.

The paper is organized as follows. In the next section we introduce necessary definitions and formulate the main results. In Section 3 some facts about transformation operators are presented. Proof of Theorem 2.1 and a counterexample are given in Section 4, and Theorems 2.2 and 2.3 are proved in Section 5. Finally, Appendix A contains the proof of Theorem 3.4.

## 2. FORMULATION OF THE MAIN RESULTS

Throughout the paper we shall denote by  $(\cdot, \cdot)$  and  $\|\cdot\|$  respectively the scalar product and the norm in the Hilbert space  $L_2(0, 1)$ . The symbol  $W_p^s(a, b)$ ,  $p \geq 1$ ,  $s \in \mathbb{R}$ , shall stand for the Sobolev function space over  $(a, b)$ , and we write  $\text{Re } L_2(a, b)$  and  $\text{Re } W_2^1(a, b)$  for the real Hilbert spaces of real-valued functions from  $L_2(a, b)$  and  $W_2^1(a, b)$  respectively.

Suppose that  $q$  is a real-valued distribution from  $W_2^{-1}(0, 1)$  and that  $\sigma \in L_2(0, 1)$  is any of its (real-valued) distributional primitive. Then the differential expression  $\ell$  of (1.1) can be written as

$$\ell_\sigma(y) = -(y' - \sigma y)' - \sigma y' =: -(y^{[1]})' - \sigma y'.$$

In what follows, the symbol  $y^{[1]}$  will stand for the *quasi-derivative*  $y' - \sigma y$  of a function  $y \in W_1^1(0, 1)$ . A natural  $L_2$ -domain of  $\ell_\sigma$  is

$$\text{dom } \ell_\sigma = \{y \in W_1^1(0, 1) \mid y^{[1]} \in W_1^1(0, 1), \ell_\sigma(y) \in L_2(0, 1)\}.$$

For any  $h_0, h_1 \in \mathbb{R} \cup \{\infty\}$  we denote by  $T = T(\sigma, h_0, h_1)$  an operator in  $L_2(0, 1)$  that acts as  $Ty = \ell_\sigma(y)$  on the domain

$$\text{dom } T = \{y \in \text{dom } \ell_\sigma \mid y^{[1]}(0) = h_0 y(0), y^{[1]}(1) = h_1 y(1)\}.$$

This method of regularization by quasi-derivatives was suggested by Shkalikov and Savchuk in [25] (see also [26]). Since  $\ell_\sigma(y) = -y'' + \sigma'y$  in the sense of distributions, so defined operator  $T$  can be regarded as a Sturm–Liouville operator with the singular potential  $q = \sigma' \in W_2^{-1}(0, 1)$ . It is known [25] that  $T$  is a selfadjoint bounded below operator with discrete spectrum.

Note that  $T(\sigma + h, h_0, h_1) = T(\sigma, h_0 + h, h_1 + h)$  for any  $h \in \mathbb{R}$ ; henceforth, by adding a suitable constant  $h$  to the primitive  $\sigma$ , we can always achieve one of the

following situations: (1)  $h_0 = 0$ ,  $h_1 \in \mathbb{R}$ ; (2)  $h_0 = 0$ ,  $h_1 = \infty$ , (3)  $h_0 = \infty$ ,  $h_1 = 0$ , (4)  $h_0 = h_1 = \infty$ . All four cases are treated in a similar way, and, to be definite, we shall concentrate on the first case here.

Therefore we shall assume that the operator  $T$  has the form  $T(\sigma, 0, h) =: T_{\sigma, h}$  for some  $\sigma \in \text{Re } L_2(0, 1)$  and some  $h \in \mathbb{R}$ . As was mentioned above,  $T$  so defined is selfadjoint, bounded below, and has simple discrete spectrum accumulating at  $+\infty$ . By adding a suitable constant  $C$  to the potential  $q$  (or a suitable linear function  $Cx$  to the primitive  $\sigma$ ) if necessary, we can make all the eigenvalues  $\lambda_n^2$ ,  $n \in \mathbb{Z}_+$ , of  $T$  positive and will tacitly assume this in what follows. It is known [24, 14] that the eigenvalues  $\lambda_n^2$ , when arranged in increasing order, obey the following asymptotic formula:

$$\lambda_n = \pi n + \mu_n, \quad n \in \mathbb{Z}_+,$$

where  $(\mu_n)_{n \in \mathbb{Z}_+} \in \ell_2$ .

We denote by  $\mathfrak{L}$  the set of all strictly increasing sequences  $\Lambda = (\lambda_n)_{n \in \mathbb{Z}_+}$ , in which  $\lambda_n$  are positive numbers such that  $\mu_n := \lambda_n - \pi n$  form an  $\ell_2$ -sequence. Fix an arbitrary  $\Lambda = (\lambda_n) \in \mathfrak{L}$  and denote by  $\Pi_\Lambda$  the set of all real-valued functions  $\psi \in L_2(0, 1)$  of the form

$$(2.1) \quad \psi(x) = \sum_{n=0}^{\infty} [\alpha_n \cos(\lambda_n x) - \cos(\pi n x)] + \frac{1}{2},$$

where  $(\alpha_n)_{n \in \mathbb{Z}_+}$  is a sequence of positive numbers such that the sequence  $(\alpha_n - 1)_{n \in \mathbb{Z}_+}$  belongs to  $\ell_2$ . Put

$$(2.2) \quad \psi_\Lambda(x) := \sum_{n=0}^{\infty} [\cos(\lambda_n x) - \cos(\pi n x)];$$

it can be shown (see Lemma 4.3) that this series converges in  $L_2(0, 1)$ . Since the system  $\{\cos(\lambda_n x)\}_{n=0}^{\infty}$  forms a Riesz basis of  $L_2(0, 1)$  [12], we conclude that for any  $\psi \in L_2(0, 1)$  there is an  $\ell_2$ -sequence  $(\beta_n)$  such that

$$\psi(x) - \psi_\Lambda(x) - \frac{1}{2} = \sum_{n=0}^{\infty} \beta_n \cos \lambda_n x.$$

It follows that  $\psi$  admits representation (2.1) with  $\alpha_n := \beta_n + 1$ . Thus the only restriction imposed by  $\Pi_\Lambda$  is that all the coefficients  $\beta_n$  in the above series representation of  $\psi - \psi_\Lambda - \frac{1}{2}$  should be greater than  $-1$ . The above arguments also show that  $\Pi_\Lambda$  is an open and convex set in  $\text{Re } L_2(0, 1)$ .

Assume that  $\sigma_0 \in \text{Re } L_2(0, \frac{1}{2})$  and denote by  $y_0(\cdot, \lambda) = y_0(\cdot, \lambda, \sigma_0)$ ,  $\lambda \in \mathbb{C}$ , the solution of the equation

$$-(y' - \sigma_0 y)' - \sigma_0 y' = \lambda^2 y$$

on the interval  $(0, \frac{1}{2})$  subject to the initial conditions

$$y(0) = 1, \quad (y' - \sigma_0 y)(0) = 0.$$

Let  $l_{0, \sigma_0}(x, t)$  be the kernel of the transformation operator  $I + L_{0, \sigma_0}$  that maps  $y_0(\cdot, \lambda)$  into the function  $\cos \lambda x$  for all complex  $\lambda$  (see Section 3 for details). In other words,  $l_{0, \sigma_0}$  is such that the following equality is satisfied for all  $\lambda \in \mathbb{C}$ ,  $x \in (0, \frac{1}{2})$ :

$$\cos \lambda x = y_0(x, \lambda) + \int_0^x l_{0, \sigma_0}(x, t) y_0(t, \lambda) dt.$$

We put

$$(2.3) \quad \phi_0(2x) = \phi_0(2x, \sigma_0) := -\frac{1}{2}\sigma_0(x) + \int_0^x l_{0,\sigma_0}^2(x, t) dt, \quad x \in (0, \tfrac{1}{2}).$$

It follows from the results of [14] that  $\phi_0 \in L_2(0, 1)$ ; see also Remark 3.5.

Our main result is the following

**Theorem 2.1.** *Assume that  $\Lambda = (\lambda_n)_{n \in \mathbb{Z}_+} \in \mathfrak{L}$ ,  $\sigma_0 \in \text{Re } L_2(0, \frac{1}{2})$ , and define  $\phi_0 := \phi_0(\cdot, \sigma_0)$  as in (2.3).*

- (i) *The half-inverse spectral problem is soluble for the mixed spectral data  $\{\sigma_0, \Lambda\}$  if and only if the function  $\phi_0$  belongs to  $\Pi_\Lambda$ .*
- (ii) *If  $\phi_0 \in \Pi_\Lambda$ , then the solution of the above half-inverse spectral problem is unique—i.e., there exist a unique  $\sigma \in \text{Re } L_2(0, 1)$  and a unique  $h \in \mathbb{R}$  such that  $\sigma$  is an extension of  $\sigma_0$  and the spectrum of  $T_{\sigma,h}$  coincides with  $\Lambda^2 := (\lambda_n^2)_{n \in \mathbb{Z}_+}$ .*

As the function  $\phi_0$  is determined via  $\sigma_0$  and the set  $\Pi_\Lambda$  via  $\Lambda$ , condition (i) of the theorem imposes connection between  $\sigma_0$  and  $\Lambda$ . An example of mixed data  $\{\sigma_0, \Lambda\}$ , for which  $\phi_0$  is not in  $\Pi_\Lambda$  (and thus the half-inverse problem has no solution), is given at the end of Section 4.

For the “unperturbed” situation with zero potential one has  $\sigma_0 \equiv 0$  on  $(0, \frac{1}{2})$ ,  $\lambda_n = \pi n$ ,  $n \in \mathbb{Z}_+$ , and  $\phi_0 \equiv \frac{1}{2}$ , so that  $\phi_0 \in \Pi_\Lambda$ . Since the set  $\Pi_\Lambda$  is open and depends continuously on  $\Lambda$ , and the function  $\phi_0$  of (2.3) depends continuously on  $\sigma_0$ , it follows that  $\phi_0 \in \Pi_\Lambda$  if the function  $\sigma_0 \in \text{Re } L_2(0, \frac{1}{2})$  and the sequence  $\Lambda - (\pi n) \in \ell_2$  have small norms. This is a local existence result analogous to those of the papers [1, 23]. However, nice bounds on the norms  $\|\sigma_0\|_{L_2(0,1/2)}$  and  $\|\Lambda - (\pi n)\|_{\ell_2}$  are cumbersome and difficult to obtain. Instead, we shall establish such bounds in the particular case where  $\sigma_0 \in W_2^1(0, \frac{1}{2})$ , i.e., where  $q_0 \in L_2(0, \frac{1}{2})$ , and estimate the norm of  $q_0$  in  $L_2(0, \frac{1}{2})$ .

**Theorem 2.2.** *Assume that a real-valued function  $q_0 \in L_2(0, \frac{1}{2})$  and a sequence  $(\lambda_n) =: \Lambda \in \mathfrak{L}$  are such that  $\|q_0\|_{L_2(0,1/2)} \leq \frac{1}{2}$  and  $\|(\lambda_n - \pi n)\|_{\ell_2} \leq \frac{1}{4}$ . Then  $\phi_0 \in \Pi_\Lambda$ ; therefore there exists a unique function  $\sigma \in \text{Re } L_2(0, 1)$  extending  $\sigma_0 := \int_0^x q_0$  and a unique real  $h$  such that the numbers  $\lambda_n^2$  are eigenvalues of the Sturm–Liouville operator  $T_{\sigma,h}$ .*

Theorems 2.1 and 2.2 admit the following refinement for the class of regular Sturm–Liouville operators with potentials from  $L_2(0, 1)$ . We observe that if the potential  $q$  belongs to  $L_2(0, 1)$  and  $\sigma(x) = \int_0^x q(t) dt$ , then the operator  $T = T_{\sigma,h}$  is given by

$$Ty := -y'' + qy, \quad \text{dom } T = \{y \in W_2^2(0, 1) \mid y'(0) = 0, y'(1) = \hat{h}y(1)\}$$

with  $\hat{h} := h + \sigma(1)$ , and the eigenvalues  $\lambda_n^2$  of  $T$  obey the asymptotics

$$\lambda_n = \pi n + \frac{c}{n} + \frac{\nu_n}{n}, \quad n \in \mathbb{Z}_+,$$

for some  $c \in \mathbb{R}$  and an  $\ell_2$ -sequence  $(\nu_n)$ . We denote by  $\mathfrak{L}_1$  a subset of  $\mathfrak{L}$  formed by sequences  $(\lambda_n)$  obeying this refined asymptotics.

**Theorem 2.3.** *Assume that  $\Lambda \in \mathfrak{L}_1$  and  $\sigma_0 \in \text{Re } W_2^1(0, \frac{1}{2})$ . If  $\phi_0 \in \Pi_\Lambda$ , then the extended function  $\sigma$  given by Theorem 2.1 belongs to  $W_2^1(0, 1)$ .*

In other words, this theorem states that if for a given function  $q_0 \in \text{Re } L_2(0, \frac{1}{2})$  and a given sequence  $(\lambda_n) \in \mathfrak{L}_1$  the half-inverse spectral problem has a solution within the class of Sturm–Liouville operators with potentials from  $W_2^{-1}(0, 1)$ , then the recovered potential belongs in fact to  $L_2(0, 1)$ .

## 3. TRANSFORMATION OPERATORS

In this section we shall formulate some results from the papers [14, 15] that will be used later on to establish our principal results.

Suppose that  $\sigma \in L_2(0, 1)$  and denote by  $\tilde{T}_\sigma$  an operator in  $L_2(0, 1)$  that acts according to

$$\tilde{T}_\sigma y = \ell_\sigma(y) := -(y^{[1]})' - \sigma y'$$

on the domain

$$\text{dom } \tilde{T}_\sigma = \{y \in \text{dom } \ell_\sigma \mid y^{[1]}(0) = 0\}.$$

In other words,  $\tilde{T}_\sigma$  is an extension of the operator  $T_{\sigma,h}$  discarding the boundary condition at the terminal point  $x = 1$ .

One of the main results of the paper [14] is that the operators  $\tilde{T}_\sigma$  and  $\tilde{T}_0$  are similar, and the similarity is performed by a *transformation operator* of a special form.

**Theorem 3.1.** *Assume that  $\sigma \in L_2(0, 1)$ ; then there exists an integral Hilbert–Schmidt operator  $K_\sigma : L_2(0, 1) \rightarrow L_2(0, 1)$  of the form*

$$(3.1) \quad (K_\sigma u)(x) = \int_0^x k_\sigma(x, t) u(t) dt$$

such that  $I + K_\sigma$  is a transformation operator for  $\tilde{T}_\sigma$  and  $\tilde{T}_0$ , i.e., such that

$$(3.2) \quad \tilde{T}_\sigma(I + K_\sigma) = (I + K_\sigma)\tilde{T}_0.$$

The operator  $K_\sigma$  with properties (3.1)–(3.2) is unique. If, moreover, the function  $\sigma$  is real-valued, then the kernel  $k_\sigma$  is real-valued, too.

Put  $L_\sigma := (I + K_\sigma)^{-1} - I$ ; then  $L_\sigma$  is an integral Hilbert–Schmidt operator of Volterra type, i.e.,

$$(L_\sigma u)(x) = \int_0^x l_\sigma(x, t) u(t) dt,$$

and  $I + L_\sigma$  is the transformation operator for  $\tilde{T}_0$  and  $\tilde{T}_\sigma$ . The kernels  $k_\sigma$  and  $l_\sigma$  of the operators  $K_\sigma$  and  $L_\sigma$  possess the property that, for any fixed  $x \in [0, 1]$ , the functions  $k_\sigma(x, \cdot)$  and  $l_\sigma(x, \cdot)$  belong to  $L_2(0, 1)$  and the mappings

$$\begin{aligned} [0, 1] \ni x &\mapsto k_\sigma(x, \cdot) \in L_2(0, 1), \\ [0, 1] \ni x &\mapsto l_\sigma(x, \cdot) \in L_2(0, 1) \end{aligned}$$

are continuous.

The transformation operators naturally appear during factorization of some Fredholm operators, which we shall now explain.

Denote by  $\mathfrak{S}_2$  the ideal of all Hilbert–Schmidt operators in  $L_2(0, 1)$ . It is known that any operator in  $\mathfrak{S}_2$  is an integral operator with square integrable kernel on  $(0, 1) \times (0, 1)$ . We denote by  $\mathfrak{S}_2^+$  ( $\mathfrak{S}_2^-$ ) the subalgebra of  $\mathfrak{S}_2$  consisting of all integral operators in  $\mathfrak{S}_2$  with upper-diagonal (respectively, lower-diagonal) kernels.

**Definition 3.2.** Assume that  $Q \in \mathfrak{S}_2$ . We say that the operator  $I + Q$  is *factorizable*, or that  $I + Q$  *admits factorization*, if there exist operators  $R^+ \in \mathfrak{S}_2^+$  and  $R^- \in \mathfrak{S}_2^-$  such that

$$I + Q = (I + R^+)^{-1}(I + R^-)^{-1}.$$

Observe that an operator  $I + Q$  can admit at most one factorization and thus the operators  $R^\pm = R^\pm(Q)$  are uniquely determined by  $Q$ .

To every function  $\phi \in L_2(0, 2)$  we shall put into correspondence an integral operator  $F_\phi \in \mathfrak{S}_2$  with kernel  $f_\phi(x, t) := \phi(x + t) + \phi(|x - t|)$ , i.e.,

$$(F_\phi u)(x) := \int_0^1 f_\phi(x, t) u(t) dt.$$

Denote by  $\Phi$  the set of those  $\phi \in L_2(0, 2)$ , for which the corresponding operator  $F_\phi$  admits factorization. It follows from the results of [21] that the set  $\Phi$  is open and everywhere dense in  $L_2(0, 2)$ . We note also that if  $F_\phi$  is a selfadjoint operator, then  $\phi \in \Phi$  if and only if  $I + F_\phi$  is (uniformly) positive in  $L_2(0, 1)$ , see [9, Ch. 4].

Connection between the transformation operators  $K_\sigma$  and operators  $F_\phi$  is described by the following statement, cf. [14].

**Theorem 3.3.** (i) Let  $\sigma \in L_2(0, 1)$  and define a function  $\phi = \phi_\sigma$  via

$$(3.3) \quad \phi(2x) = -\frac{1}{2}\sigma(x) + \int_0^x l_\sigma^2(x, t) dt, \quad x \in (0, 1).$$

Then  $\phi \in \Phi$  and

$$(3.4) \quad I + F_\phi = (I + K_\sigma)^{-1}(I + K_\sigma^\top)^{-1},$$

where  $I + K_\sigma$  is the transformation operator for  $\tilde{T}_\sigma$  and  $\tilde{T}_0$ , and  $K_\sigma^\top$  is the operator associated to  $K_\sigma$ , i.e.,

$$(K_\sigma^\top u)(x) := \int_x^1 k_\sigma(t, x) u(t) dt.$$

- (ii) Conversely, if  $\phi \in \Phi$  and  $I + F_\phi = (I + K^+)^{-1}(I + K^-)^{-1}$  with  $K^\pm \in \mathfrak{S}_2^\pm$ , then  $K = K_\sigma$  for some  $\sigma \in L_2(0, 1)$ ,  $K^- = K_\sigma^\top$ , and (3.3) holds.
- (iii) The mapping  $L_2(0, 1) \ni \sigma \mapsto \phi \in \Phi$  given by (3.3) is homeomorphic.

The same statements hold certainly true if we consider the Sturm–Liouville problem on the interval  $(0, \frac{1}{2})$  instead of  $(0, 1)$ ; the function  $\phi$  will then be defined on  $(0, 1)$  instead of  $(0, 2)$ .

It can be proved that for a smooth function  $\sigma$  the function  $\phi$  of (3.3) is also smooth. We shall need the following version of this statement (proved in Appendix A).

**Theorem 3.4.** The restriction of the mapping  $L_2(0, 1) \ni \sigma \mapsto \phi \in \Phi$  given by (3.3) to the set  $W_2^1(0, 1)$  is a bijection onto  $\Phi \cap W_2^1(0, 2)$ .

Assume that  $h \in \mathbb{R}$ ,  $\sigma \in \text{Re } L_2(0, 1)$ , and let  $(\lambda_n^2)_{n=0}^\infty$  be the sequence of eigenvalues of the operator  $T = T_{\sigma, h}$ . Denote by  $u_n$  the eigenfunction of  $T$  corresponding to the eigenvalue  $\lambda_n^2$  and normalized by the initial condition  $u_n(0) = \sqrt{2}$ . Then  $u_n = (I + K_\sigma)v_n$ , where  $v_n(x) := \sqrt{2} \cos(\lambda_n x)$ . Put

$$\alpha_n := \|u_n\|_{L_2(0, 1)}^{-2};$$

then the asymptotics of  $\lambda_n$  and properties of the transformation operator  $I + K_\sigma$  imply that

$$\alpha_n = 1 + \beta_n,$$

where the sequence  $(\beta_n)_{n \in \mathbb{Z}_+}$  belongs to  $\ell_2(\mathbb{Z}_+)$  [15]. Using the Parseval identity

$$\text{s-lim}_{N \rightarrow \infty} \sum_{n=0}^N \alpha_n(\cdot, u_n) u_n = I$$

(s-lim denoting the limit in the strong operator topology in  $L_2(0, 1)$ ), replacing  $u_n$  by  $(I + K_\sigma)v_n$ , and recalling relation (3.4), we conclude that

$$I + F_\phi = \text{s-lim}_{N \rightarrow \infty} \sum_{n=0}^N \alpha_n(\cdot, v_n)v_n.$$

Straightforward calculations show that the function  $\phi$  of (3.3) determining the kernel  $f_\phi$  of the operator  $F_\phi$  is given by the series

$$(3.5) \quad \phi(x) = \sum_{n=0}^{\infty} [\alpha_n \cos(\lambda_n x) - \cos(\pi n x)] + \frac{1}{2},$$

the equality being understood in the  $L_2(0, 2)$ -sense.

**Remark 3.5.** Assume that  $\sigma_0 \in L_2(0, \frac{1}{2})$  and that  $\sigma$  is an arbitrary extension of  $\sigma_0$  to a function from  $L_2(0, 1)$ . It is easily seen that the restriction of the kernel  $l_\sigma$  onto the square  $(0, \frac{1}{2}) \times (0, \frac{1}{2})$  coincides with the kernel  $l_{0, \sigma_0}$  of the transformation operator related to the function  $\sigma_0$ , see Section 2. Thus the function  $\phi_0$  given by formula (2.3) verifies the equality

$$\phi_0(2x) = -\frac{1}{2}\sigma(x) + \int_0^x l_\sigma^2(x, t) dt, \quad x \in (0, \frac{1}{2});$$

in particular,  $\phi_0$  is the restriction to  $(0, 1)$  of the function  $\phi = \phi_\sigma$  of (3.3), which implies the inclusion  $\phi_0 \in L_2(0, 1)$ .

#### 4. PROOF OF THEOREM 2.1

We start with establishing several lemmata.

**Lemma 4.1.** Assume that  $(\lambda_n)_{n \in \mathbb{Z}_+} \in \mathfrak{L}$  and  $(\beta_n)_{n \in \mathbb{Z}_+} \in \ell_2$ . Then the series

$$\sum_{n=0}^{\infty} \beta_n \cos(\lambda_n x) \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n \sin(\lambda_n x)$$

converge in  $L_2(0, 2)$ .

*Proof.* We observe that the systems  $\{\cos(\lambda_n x)\}_{n \in \mathbb{Z}_+}$  and  $\{\sin(\lambda_n x)\}_{n \in \mathbb{N}}$  form Riesz bases of  $L_2(0, 1)$  [12]. Convergence of both series in  $L_2(0, 1)$  now follows from the definition of a Riesz basis, see [8, Ch. VI]. Since

$$\beta_n \cos[\lambda_n(x+1)] = \beta_n \cos \lambda_n \cos(\lambda_n x) - \beta_n \sin \lambda_n \sin(\lambda_n x)$$

and the sequences  $(\beta_n \cos \lambda_n)$  and  $(\beta_n \sin \lambda_n)$  belong to  $\ell_2(\mathbb{Z}_+)$ , the series

$$\sum_{n=0}^{\infty} \beta_n \cos[\lambda_n(x+1)] = \sum_{n=0}^{\infty} \beta_n \cos \lambda_n \cos(\lambda_n x) - \sum_{n=0}^{\infty} \beta_n \sin \lambda_n \sin(\lambda_n x)$$

also converges in  $L_2(0, 1)$ . Henceforth the series  $\sum_{n=0}^{\infty} \beta_n \cos(\lambda_n x)$  converges in  $L_2(0, 2)$ . The second series is treated in the same manner, and the lemma is proved.  $\square$

**Lemma 4.2.** For any real numbers  $a$  and  $b$  the following inequality holds:

$$|\cos(a+b) - \cos a + b \sin a| \leq \frac{b^2}{\sqrt{3}}.$$



*Proof.* Using the standard trigonometric identities, we find that

$$\begin{aligned} |\cos(a+b) - \cos a + b \sin a| &= |(\cos b - 1) \cos a + (b - \sin b) \sin a| \\ &\leq \sqrt{(\cos b - 1)^2 + (b - \sin b)^2} \\ &= \sqrt{2(1 - \cos b) + b(b - 2 \sin b)}. \end{aligned}$$

Taking into account the inequalities  $1 - \cos b \leq b^2/2$  and  $|b - \sin b| \leq |b|^3/6$ , holding for all real  $b$ , we conclude that

$$2(1 - \cos b) + b(b - 2 \sin b) \leq 2b(b - \sin b) \leq b^4/3,$$

and the result follows.  $\square$

**Lemma 4.3.** *Assume that  $(\lambda_n) \in \mathfrak{L}$  and  $(\nu_n) \in \ell_\infty(\mathbb{Z}_+)$ ; then the series*

$$(4.1) \quad \sum_{n=0}^{\infty} \nu_n [\cos(\lambda_n x) - \cos(\pi n x)] =: \theta(x)$$

*converges in  $L_2(0, 2)$ . Moreover, putting  $\mu_n := \lambda_n - \pi n$  and denoting  $\gamma := \|(\mu_n)\|_{\ell_2}$ ,  $\delta := \|(\nu_n)\|_{\ell_\infty}$  we have*

$$\|\theta\|_{L_2(0,1)} \leq \delta \left( \frac{\gamma^2}{\sqrt{15}} + \frac{\gamma}{\sqrt{2}} \right).$$

*Proof.* By the definition of the set  $\mathfrak{L}$ , the sequence  $(\mu_n)_{\mathbb{Z}_+}$  falls into  $\ell_2$ . Denote

$$\tilde{\theta}(x) := -x \sum_{n=0}^{\infty} \mu_n \nu_n \sin(\pi n x);$$

then  $\tilde{\theta} \in L_2(0, 2)$  by Lemma 4.1 and, since the set  $\{\sqrt{2} \sin(\pi n x)\}_{n \in \mathbb{N}}$  is an orthonormal basis of  $L_2(0, 1)$ ,

$$\|\tilde{\theta}\|_{L_2(0,1)} \leq \left( \frac{1}{2} \sum_{n=0}^{\infty} |\mu_n \nu_n|^2 \right)^{1/2} \leq \frac{\gamma \delta}{\sqrt{2}}.$$

Applying Lemma 4.2, we find that

$$(4.2) \quad |\cos(\lambda_n x) - \cos(\pi n x) + \mu_n x \sin(\pi n x)| \leq \frac{|\mu_n x|^2}{\sqrt{3}}, \quad x \in [0, 2],$$

so that the series

$$\sum_{n=0}^{\infty} \nu_n [\cos(\lambda_n x) - \cos(\pi n x) + \mu_n x \sin(\pi n x)]$$

converges uniformly and absolutely on  $[0, 2]$ . This proves that series (4.1) converges in  $L_2(0, 2)$ ; we denote its sum by  $\theta$ .

Inequality (4.2) yields the estimate

$$|\theta(x) - \tilde{\theta}(x)| = \left| \sum_{n=0}^{\infty} \nu_n [\cos(\lambda_n x) - \cos(\pi n x) + \mu_n x \sin(\pi n x)] \right| \leq \frac{\gamma^2 \delta}{\sqrt{3}} x^2,$$

so that  $\|\theta - \tilde{\theta}\|_{L_2(0,1)} \leq \gamma^2 \delta / \sqrt{15}$  and

$$\|\theta\|_{L_2(0,1)} \leq \|\theta - \tilde{\theta}\|_{L_2(0,1)} + \|\tilde{\theta}\|_{L_2(0,1)} \leq \frac{\gamma^2 \delta}{\sqrt{15}} + \frac{\gamma \delta}{\sqrt{2}}$$

as claimed. The proof is complete.  $\square$

**Lemma 4.4.** Assume that  $(\lambda_n) \in \mathfrak{L}$  and that  $(\alpha_n)$  is a sequence of real numbers such that  $c < \alpha_n < C$  for some positive constants  $c, C$  and all  $n \in \mathbb{Z}_+$ . Define an operator  $U : L_2(0, 1) \rightarrow L_2(0, 1)$  by the equality

$$(4.3) \quad U := \text{s-lim}_{N \rightarrow \infty} \sum_{n=0}^N \alpha_n(\cdot, v_n) v_n,$$

where s-lim stands for the limit in the strong operator topology of  $L_2(0, 1)$  and  $v_n(x) := \sqrt{2} \cos(\lambda_n x)$ . Then the operator  $U$  is invertible and  $(U^{-1}v_j, v_k) = \alpha_k^{-1} \delta_{jk}$  for all  $j, k \in \mathbb{Z}_+$ , where  $\delta_{jk}$  is the Kronecker delta.

*Proof.* Observe that in view of the relation  $\lambda_n - \pi n \rightarrow 0$  as  $n \rightarrow \infty$  the system  $\{\cos(\lambda_n x)\}_{n \in \mathbb{Z}_+}$  is a Riesz basis of  $L_2(0, 1)$  [12], which yields convergence of the sum in (4.3) in the strong operator topology. It is easily seen that

$$U^{-1} = \text{s-lim}_{N \rightarrow \infty} \sum_{n=1}^N \alpha_n^{-1}(\cdot, \hat{v}_n) \hat{v}_n,$$

where  $(\hat{v}_n)$  is a basis biorthogonal to  $(v_n)$ , see [8, Ch. VI]. Therefore

$$(U^{-1}v_j, v_k) = \text{s-lim}_{N \rightarrow \infty} \sum_{l=1}^N \alpha_l^{-1}(v_j, \hat{v}_l)(\hat{v}_l, v_k) = \alpha_k^{-1} \delta_{jk} = \alpha_k^{-1} \delta_{jk},$$

and the proof is complete.  $\square$

*Proof of Theorem 2.1.* (i) *Necessity.* Assume that  $\sigma_0 \in \text{Re } L_2(0, \frac{1}{2})$ ,  $(\lambda_n) =: \Lambda \in \mathfrak{L}$ , and let there exist an extension  $\sigma \in L_2(0, 1)$  of  $\sigma_0$  to  $(0, 1)$  and a number  $h \in \mathbb{R}$  such that the spectrum of the corresponding operator  $T_{\sigma, h}$  coincides with the set  $\{\lambda_n^2\}$ . Then according to Remark 3.5 and formula (3.5) we have

$$(4.4) \quad \phi_0(x) = \sum_{n=0}^{\infty} [\alpha_n \cos(\lambda_n x) - \cos(\pi n x)] + \frac{1}{2}, \quad x \in (0, 1)$$

for some  $\alpha_n > 0$  such that  $\beta_n := \alpha_n - 1$  form an  $\ell_2$ -sequence, so that  $\phi_0 \in \Pi_\Lambda$  as required.

*Sufficiency.* Assume that  $\sigma_0 \in \text{Re } L_2(0, \frac{1}{2})$ ,  $(\lambda_n) =: \Lambda \in \mathfrak{L}$ , and that the function  $\phi_0$  belongs to  $\Pi_\Lambda$ . According to the definition of the set  $\Pi_\Lambda$ , the function  $\phi_0$  has the form (4.4) for given  $\lambda_n$  and some sequence  $(\alpha_n)$  of positive numbers, for which  $\beta_n := \alpha_n - 1$  form an  $\ell_2$ -sequence. Writing  $\alpha_n \cos(\lambda_n x) - \cos(\pi n x)$  as

$$\beta_n \cos(\lambda_n x) + [\cos(\lambda_n x) - \cos(\pi n x)]$$

and applying Lemmata 4.1 and 4.3, we conclude that the series on the right-hand side of (4.4) converges in  $L_2(0, 2)$  to some function  $\phi$ ; clearly,  $\phi(x) = \phi_0(x)$  a.e. on  $(0, 1)$ .

Consider now the operator  $I + F_\phi$  corresponding to the function  $\phi$  constructed above. Using the definition of  $F_\phi$ , one easily shows that

$$I + F_\phi = \text{s-lim}_{N \rightarrow \infty} \sum_{n=0}^N \alpha_n(\cdot, v_n) v_n$$

with  $v_n(x) = \sqrt{2} \cos(\lambda_n x)$ . Since the set  $\{v_n\}_{n \in \mathbb{Z}_+}$  forms a Riesz basis of  $L_2(0, 1)$  and the numbers  $\alpha_n$  are uniformly positive and uniformly bounded, the operator  $I + F_\phi$  is

bounded and (uniformly) positive. It follows that the operator  $I + F_\phi$  is factorizable [9, Ch. 4], so that  $\phi \in \Phi$  and by Theorem 3.3 there exists  $\sigma \in L_2(0, 1)$  such that

$$(4.5) \quad \phi(2x) = -\frac{1}{2}\sigma(x) + \int_0^x l_\sigma^2(x, t) dt, \quad x \in (0, 1).$$

Here  $l_\sigma$  is the kernel of the transformation operator corresponding to  $\tilde{T}_0$  and  $\tilde{T}_\sigma$ .

We prove next that  $\sigma = \sigma_0$  on  $(0, \frac{1}{2})$ . Denote by  $\tilde{\sigma}_0$  the restriction of  $\sigma$  to  $(0, \frac{1}{2})$ ; then by Remark 3.5

$$l_\sigma(x, t) = l_{0, \tilde{\sigma}_0}(x, t), \quad 0 < t < x < \frac{1}{2},$$

where  $l_{0, \tilde{\sigma}_0}$  is the kernel of the transformation operator on the interval  $(0, \frac{1}{2})$  constructed for the function  $\tilde{\sigma}_0$ . On the other hand,  $\phi_0$  satisfies the relation

$$\phi_0(2x) = -\frac{1}{2}\sigma_0(x) + \int_0^x l_{0, \sigma_0}^2(x, t) dt, \quad x \in (0, \frac{1}{2}),$$

and since the mapping (3.3) considered on  $(0, \frac{1}{2})$  is bijective, we conclude that  $\tilde{\sigma}_0 = \sigma_0$ , i.e., that  $\sigma$  is an extension of  $\sigma_0$ .

It remains to show that there is  $h \in \mathbb{R}$  such that the spectrum of  $T_{\sigma, h}$  coincides with the set  $\{\lambda_n^2\}_{n \in \mathbb{Z}_+}$ . Put

$$w_j := (I + K_\sigma)v_j, \quad j \in \mathbb{Z}_+;$$

then by virtue of Lemma 4.4 we find that

$$(w_j, w_k) = ((I + K_\sigma^*)(I + K_\sigma)v_j, v_k) = ((I + F_\phi)^{-1}v_j, v_k) = \alpha_k^{-1}\delta_{jk}.$$

In particular, the vectors  $w_j$  are orthogonal; since  $\{v_k\}_{k \in \mathbb{Z}_+}$  is a Riesz basis of  $L_2(0, 1)$  and the operator  $I + K_\sigma$  is a homeomorphism of  $L_2(0, 1)$ , the set  $\{w_j\}_{j \in \mathbb{Z}_+}$  is an orthogonal Riesz basis of  $L_2(0, 1)$ .

It remains to prove that there is  $h \in \mathbb{R}$  such that  $w_j$  are eigenfunctions of the operator  $T_{\sigma, h}$ , i.e., that  $w_j^{[1]} = hw_j(1)$  for all  $j \in \mathbb{Z}_+$ . We observe that  $w_j$  satisfy the following Lagrange identity:

$$(4.6) \quad 0 = (\tilde{T}_\sigma w_j, w_k) - (w_j, \tilde{T}_\sigma w_k) = -w_j^{[1]}(1)\overline{w_k(1)} + w_j(1)\overline{w_k^{[1]}(1)}.$$

If  $w_j(1) = 0$  for some  $j \in \mathbb{Z}_+$ , then  $w_j^{[1]}(1) \neq 0$  by uniqueness theorem for the equation  $\ell_\sigma(y) = \lambda_j^2 y$  [25] and the above relations imply that  $w_k(1) = 0$  for all  $k \in \mathbb{Z}_+$ . Then  $\lambda_k^2$  would be eigenvalues of the Sturm–Liouville operator  $T_{\sigma, \infty}$ . This is impossible since the eigenvalues  $\nu_k^2$  of  $T_{\sigma, \infty}$  obey the asymptotics  $\nu_k = \pi(n - \frac{1}{2}) + o(1)$  as  $k \rightarrow \infty$  [15, 26], and this asymptotics is completely different from that of  $\lambda_k$ . Henceforth  $w_k(1) \neq 0$  for all  $k \in \mathbb{Z}_+$ , so that by (4.6) there exists  $h \in \mathbb{R}$  such that  $w_k^{[1]}(1)/w_k(1) = h$ , i.e., such that the set  $\{\lambda_k^2\}$  are eigenvalues of the operator  $T_{\sigma, h}$  and  $w_k$  are the corresponding eigenfunctions. The operator  $T_{\sigma, h}$  has no other eigenvalues since the set  $\{w_j\}_{j \in \mathbb{Z}_+}$  of the eigenfunctions is already complete in  $L_2(0, 1)$ .

To prove (ii) we observe that the spectral data  $(\sigma_0, \Lambda)$  uniquely determine the transformation operator  $K$ , so that  $\sigma' = q$  is unique. The above reasonings show that the number  $h$  in the boundary condition is then identified as  $w_0^{[1]}(1)/w_0(1) = h$ , where  $w_0 = (I + K)v_0$  and  $v_0 = \sqrt{2}\cos(\lambda_0 x)$ . Thus the extension  $q$  of  $q_0$  and  $h \in \mathbb{R}$  are unique, and the proof is complete.  $\square$

Next we give an example showing that the inclusion  $\phi_0 \in \Pi_\Lambda$  need not hold in general, so that all the hypotheses of Theorem 2.1 are essential for solvability of the half-inverse problem.

**Example 4.5.** Set  $\lambda_n := \pi n$ ,  $n \in \mathbb{Z}_+$ ,  $\Lambda := (\lambda_n)_{n \in \mathbb{Z}_+}$ , and

$$(4.7) \quad \sigma_{\gamma,0}(x) := \frac{2\gamma}{1-\gamma x} - \gamma, \quad x \in [0, \tfrac{1}{2}].$$

Observe that  $\sigma_{\gamma,0}$  belongs to  $L_2(0, \frac{1}{2})$  if  $\gamma < 2$ . We shall show that the corresponding function  $\phi_{\gamma,0} := \phi_0(\cdot, \sigma_{\gamma,0})$  of (2.3) equals  $-\gamma/2$ ; therefore  $\phi_{\gamma,0} \in \Pi_\Lambda$  if and only if  $\gamma < 1$ , so that for  $\gamma < 2$  the half-inverse spectral problem with mixed data  $\{\sigma_{\gamma,0}, \Lambda\}$  is soluble if and only if  $\gamma < 1$ .

By a straightforward verification one sees that a solution  $y_0(\cdot, \lambda)$  to the equation

$$-(y' - \sigma_{\gamma,0}y)' - \sigma_{\gamma,0}y' = \lambda^2 y$$

satisfying the initial conditions  $y(0) = 1$ ,  $(y' - \sigma_{\gamma,0}y)(0) = 0$ , is

$$y_0(x, \lambda) = \cos(\lambda x) + \frac{\gamma \sin(\lambda x)}{\lambda(1-\gamma x)}.$$

The kernel  $l_{0,\sigma_{\gamma,0}}(x, t)$  of the transformation operator  $I + L_{0,\sigma_{\gamma,0}}$  must satisfy the following identity for all  $\lambda \in \mathbb{C}$  and all  $x \in [0, \frac{1}{2}]$ :

$$\cos(\lambda x) + \frac{\gamma \sin(\lambda x)}{\lambda(1-\gamma x)} + \int_0^x l_{0,\sigma_{\gamma,0}}(x, t) \left[ \cos(\lambda t) + \frac{\gamma \sin(\lambda t)}{\lambda(1-\gamma t)} \right] ds = \cos(\lambda x)$$

Observe that

$$\left( \frac{\gamma \sin(\lambda x)}{\lambda(1-\gamma x)} \right)' = \frac{\gamma}{1-\gamma x} \left[ \cos(\lambda x) + \frac{\gamma \sin(\lambda x)}{\lambda(1-\gamma x)} \right],$$

which suggests that

$$l_{0,\sigma_{\gamma,0}}(x, t) = -\frac{\gamma}{1-\gamma t}.$$

Now the function  $\phi_{\gamma,0}$  is found to be

$$\begin{aligned} \phi_{\gamma,0}(2x) &= -\frac{1}{2}\sigma_{\gamma,0}(x) + \int_0^x l_{0,\sigma_{\gamma,0}}^2(x, t) dt = -\frac{\gamma}{1-\gamma x} + \frac{\gamma}{2} + \int_0^x \frac{\gamma^2}{(1-\gamma t)^2} dt \\ &= -\frac{\gamma}{1-\gamma x} + \frac{\gamma}{2} + \frac{\gamma}{1-\gamma t} \Big|_{t=0}^{t=x} = -\frac{\gamma}{2}, \quad x \in (0, \tfrac{1}{2}), \end{aligned}$$

so that indeed  $\phi_{\gamma,0} \in \Pi_\Lambda$  if and only if  $\gamma < 1$ .

For  $\gamma < 1$ , the half inverse problem can be solved explicitly. Indeed, according to the reconstruction algorithm of Theorem 2.1 the extension  $\phi_\gamma$  of the function  $\phi_{\gamma,0}$  to the interval  $(0, 2)$  is given by the same formula, i.e.,  $\phi_\gamma \equiv -\gamma/2$  on  $(0, 2)$ , and the integral operator  $F_{\phi_\gamma}$  corresponding to  $\phi_\gamma$  has kernel

$$f_{\phi_\gamma}(x, t) \equiv -\gamma, \quad x, t \in [0, 1].$$

The operator  $I + F_{\phi_\gamma}$  is selfadjoint; moreover, for  $\gamma < 1$  it is uniformly positive, and thus  $\phi_\gamma \in \Phi$  for such  $\gamma$ .

A simple observation suggests that a solution  $k$  to the Gelfand–Levitan–Marchenko equation

$$k(x, t) + f_{\phi_\gamma}(x, t) + \int_0^x k(x, s)f_{\phi_\gamma}(s, t) ds = 0, \quad 0 < t < x < 1,$$

must have the form  $k(x, t) = a(x)$ ; after a straightforward calculation we conclude that

$$k(x, t) = \frac{\gamma}{1-\gamma x}, \quad 0 < t < x < 1.$$

By [14, 15]  $k$  is a kernel of the transformation operator  $K_\sigma$  with

$$\sigma(x) = 2k(x, x) + 2\phi(0) = \sigma_\gamma(x),$$

where the function  $\sigma_\gamma$  is the extension of  $\sigma_{\gamma,0}$  to the interval  $(0, 1)$  by formula (4.7). The constant  $h$  in the boundary conditions is identified as  $w_0^{[1]}(1)/w_0(1)$ , with

$$w_0(x) := (I + K_\sigma)1 = 1 + \frac{\gamma x}{1 - \gamma x} = \frac{1}{1 - \gamma x}.$$

In particular,  $w_0(1) = 1/(1 - \gamma)$ ,  $w_0'(1) = \gamma/(1 - \gamma)^2$ ,  $\sigma(1) = (\gamma + \gamma^2)/(1 - \gamma)$ , so that  $h_\gamma = -\gamma^2/(1 - \gamma)$ . Now it can be directly checked that the spectrum of the operator  $T_{\sigma_\gamma, h_\gamma}$  coincides with the set  $\{\pi^2 n^2\}_{n \in \mathbb{Z}_+}$ , the corresponding eigenfunctions being  $w_0$  above and

$$w_n(x) = (I + K_\sigma) \cos(\pi n x) = \cos(\pi n x) + \frac{\gamma}{\pi n} \frac{\sin(\pi n x)}{1 - \gamma x}, \quad n \in \mathbb{N}.$$

Thus the half-inverse problem is solved.

## 5. PROOF OF THEOREMS 2.2 AND 2.3

We proved in Theorem 2.1 that the half-inverse problem is not soluble for any mixed data  $\{\sigma_0, \Lambda\} \in \text{Re } L_2(0, \frac{1}{2}) \times \mathfrak{L}$ . However, the set of those mixed data, for which a solution exists, can easily be shown to be open in  $\text{Re } L_2(0, \frac{1}{2}) \times \mathfrak{L}$ , the topology in  $\mathfrak{L}$  being inherited from that of  $\ell_2(\mathbb{Z}_+)$  through identification of  $\Lambda = (\lambda_n) \in \mathfrak{L}$  and  $(\lambda_n - \pi n) \in \ell_2(\mathbb{Z}_+)$ . This follows from the fact that both the mapping  $\sigma_0 \mapsto \phi_0$  of (2.3) and the mapping  $\Lambda \mapsto \psi_\Lambda$  induced by (2.2) are continuous. In particular, there exists a neighbourhood in  $\text{Re } L_2(0, \frac{1}{2}) \times \mathfrak{L}$  of the “unperturbed” mixed data  $\sigma_0 \equiv 0$ ,  $\Lambda = (\pi n)_{n \in \mathbb{Z}_+}$ , in which the half-inverse problem is soluble. The aim of Theorem 2.2 is to make this observation quantitative, and to this end it suffices to estimate the norms of the functions  $\phi_0$  and  $\psi_\Lambda$  in terms of  $\sigma_0$  and  $\Lambda$ . As we mentioned in Section 2, it is easier to restrict ourselves to the case  $\sigma_0 \in \text{Re } W_2^1(0, \frac{1}{2})$ , i.e., to the case  $q_0 \in \text{Re } L_2(0, \frac{1}{2})$ , and use the corresponding norms.

We start with the following auxiliary lemma.

**Lemma 5.1.** *Assume that  $q_0 \in L_2(0, \frac{1}{2})$  is such that  $\|q_0\|_{L_2(0, 1/2)} \leq \frac{1}{2}$ . Put  $\sigma_0(x) := \int_0^x q_0(t) dt$  for  $x \in (0, \frac{1}{2})$  and construct the function  $\phi_0$  on  $(0, 1)$  via (2.3). Then  $\|\phi_0\|_{L_2(0, 1)} \leq \frac{1}{4}$ .*

*Proof.* Formula (2.3) suggests that to bound the norm of the function  $\phi_0$  we only need to estimate the kernel  $l_{0, \sigma_0}$  of the transformation operator  $L_{0, \sigma_0}$  constructed as explained in Section 2. In the case where  $q_0 \in L_2$  the differential expression  $\ell_{\sigma_0}(u) := -(u' - \sigma_0 u)' - \sigma_0 u'$  coincides with  $-u'' + q_0 u$  and it is well known (see [20, Ch. 1.2]) that the kernel  $l_0 = l_{0, \sigma_0}$  is then a unique solution to the hyperbolic equation

$$l_{xx}''(x, t) = l_{tt}''(x, t) - q_0(t)l(x, t)$$

satisfying the boundary conditions

$$l(x, x) = -\frac{1}{2} \int_0^x q_0(t) dt = -\frac{1}{2} \sigma_0(x), \quad l_t'(x, t)|_{t=0} = 0.$$

Introduce new variables  $u = x + t$  and  $v = x - t$  and put  $a(u, v) := l(\frac{u+v}{2}, \frac{u-v}{2})$ ; then standard reasonings (see, e.g., [20, Ch. 1.2]) reduce the above boundary value problem

to the integral equation

$$(5.1) \quad \begin{aligned} a(u, v) = & -\frac{1}{4} \int_v^u d\alpha \int_0^v q_0\left(\frac{\alpha - \beta}{2}\right) a(\alpha, \beta) d\beta \\ & - \frac{1}{2} \int_0^v d\alpha \int_0^\alpha q_0\left(\frac{\alpha - \beta}{2}\right) a(\alpha, \beta) d\beta \\ & - \frac{1}{2} \left[ \int_0^{u/2} q_0(\alpha) d\alpha + \int_0^{v/2} q_0(\beta) d\beta \right]. \end{aligned}$$

This integral equation possesses a unique solution  $a$ , which is continuous on the set  $0 \leq v \leq u \leq 1$ . We put

$$b(s) := \max_{0 \leq v \leq u \leq s} |a(u, v)|, \quad s \in [0, 1],$$

and observe that for any fixed  $s \in [0, 1]$  the above maximum is assumed at some point  $(u_0, v_0)$  satisfying the relation  $0 \leq v_0 \leq u_0 \leq s$ . Integral equation (5.1) then implies that

$$\begin{aligned} b(s) & \leq \frac{b(s)}{2} \int_0^s d\alpha \int_0^\alpha \left| q_0\left(\frac{\alpha - \beta}{2}\right) \right| d\beta + \int_0^{s/2} |q_0(\alpha)| d\alpha \\ & \leq [sb(s) + 1] \int_0^{s/2} |q_0(\alpha)| d\alpha \leq [b(s) + 1] \int_0^{1/2} |q_0(\alpha)| d\alpha. \end{aligned}$$

Taking into account the inequality

$$\int_0^{1/2} |q_0(\alpha)| d\alpha \leq \frac{1}{\sqrt{2}} \|q_0\|_{L_2(0,1/2)} \leq \frac{1}{2\sqrt{2}},$$

we conclude that

$$b(s) \leq \frac{1}{2\sqrt{2}} \left(1 - \frac{1}{2\sqrt{2}}\right)^{-1} = \frac{1}{2\sqrt{2} - 1} < \frac{1}{\sqrt{3}}.$$

Therefore for the solution  $l_0$  of integral equation (5.1) we find that

$$\phi_1(x) := \int_0^{x/2} l_0^2(x/2, t) dt \leq \frac{x}{6},$$

so that  $\|\phi_1\|_{L_2(0,1)} \leq \frac{1}{6\sqrt{3}}$ . Using the estimate

$$|\sigma_0(x/2)| \leq \int_0^{x/2} |q_0(\alpha)| d\alpha \leq \sqrt{\frac{x}{2}} \|q_0\|_{L_2(0,1/2)} \leq \frac{\sqrt{x}}{2\sqrt{2}},$$

we arrive at

$$\int_0^1 |\sigma_0(x/2)|^2 dx \leq \frac{1}{8} \int_0^1 x dx = \frac{1}{16}.$$

Finally, (2.3) yields the required bound

$$\|\phi_0\|_{L_2(0,1)} \leq \frac{1}{2} \|\sigma_0(\cdot/2)\|_{L_2(0,1)} + \|\phi_1\|_{L_2(0,1)} \leq \frac{1}{8} + \frac{1}{6\sqrt{3}} \leq \frac{1}{4},$$

and the proof is complete.  $\square$

*Proof of Theorem 2.2.* We have to show that under the hypotheses of the theorem the function  $\phi_0$  of (2.3) belongs to  $\Pi_\Lambda$ . As explained in Section 2, the function  $\phi_0$  can be represented in the form

$$\phi_0(x) = \sum_{n=0}^{\infty} [\alpha_n \cos(\lambda_n x) - \cos(\pi n x)] + \frac{1}{2},$$

where the real numbers  $\alpha_n$  are such that the sequence  $(\beta_n)_{n \in \mathbb{Z}_+}$  with  $\beta_n := \alpha_n - 1$  belongs to  $\ell_2$ . Thus the only thing to be proved is that all  $\alpha_n$  are positive.

Set

$$\theta(x) := \sum_{n=0}^{\infty} \alpha_n [\cos(\pi n x) - \cos(\lambda_n x)], \quad x \in (0, 1);$$

by Lemma 4.3 the above series converges in  $L_2(0, 1)$ , so that  $\theta \in L_2(0, 1)$ . Then

$$\phi_0(x) + \theta(x) = \sum_{n=0}^{\infty} (\alpha_n - 1) \cos(\pi n x) + \frac{1}{2} = \sum_{n=1}^{\infty} (\alpha_n - 1) \cos(\pi n x) + \alpha_0 - \frac{1}{2},$$

so that

$$\|\phi_0 + \theta\|_{L_2(0,1)}^2 = \frac{1}{2} \sum_{n=1}^{\infty} (\alpha_n - 1)^2 + (\alpha_0 - \frac{1}{2})^2.$$

On the other hand,  $\|\phi_0\|_{L_2(0,1)} \leq \frac{1}{4}$  by Lemma 5.1, and the norm of the function  $\theta$  can be estimated by virtue of Lemma 4.3 as

$$\|\theta\|_{L_2(0,1)} \leq \delta \left( \frac{\gamma^2}{\sqrt{15}} + \frac{\gamma}{\sqrt{2}} \right) \leq \frac{\delta}{4} \left( \frac{1}{\sqrt{2}} + \frac{1}{4\sqrt{15}} \right) \leq \frac{\delta}{4} \left( \frac{11}{15} + \frac{1}{15} \right) = \frac{\delta}{5},$$

where we have put  $\gamma := \|(\mu_n)\|_{\ell_2}$ ,  $\delta := \sup_{n \in \mathbb{Z}_+} |\alpha_n|$ , and used the fact that  $\gamma \leq \frac{1}{4}$  by the assumption of the theorem. Taking into account the above relations, we conclude that

$$(5.2) \quad \frac{1}{2} \sum_{n=1}^{\infty} (\alpha_n - 1)^2 + (\alpha_0 - \frac{1}{2})^2 \leq \left( \frac{1}{4} + \frac{\delta}{5} \right)^2,$$

which yields the estimates

$$(5.3) \quad |\alpha_0 - \frac{1}{2}| \leq \frac{1}{4} + \frac{\delta}{5}, \quad |\alpha_n - 1| \leq \sqrt{2} \left( \frac{1}{4} + \frac{\delta}{5} \right), \quad n \in \mathbb{N}.$$

These inequalities imply that

$$\delta \leq 1 + \sqrt{2} \left( \frac{1}{4} + \frac{\delta}{5} \right) \leq \frac{14}{10} + \frac{3\delta}{10},$$

i.e., that  $\delta \leq 2$ . Returning now to inequalities (5.3), we get

$$|\alpha_n - 1| \leq \frac{2\sqrt{2}}{3} < 1, \quad n \in \mathbb{N},$$

so that  $\alpha_n > 0$  for all  $n \in \mathbb{N}$ .

Assume that  $\alpha_0 < 0$ . This is only possible when  $\delta > 1$  (as otherwise  $|\alpha_0 - \frac{1}{2}| < \frac{1}{2}$  and  $\alpha_0 > 0$ ), and then  $\delta = \sup_{n \in \mathbb{N}} \alpha_n$ . Therefore relation (5.2) yields the inequality

$$\frac{1}{2}(\delta - 1)^2 + (\alpha_0 - \frac{1}{2})^2 \leq \left( \frac{1}{4} + \frac{\delta}{5} \right)^2,$$

so that, on account of the assumption  $\alpha_0 < 0$ , we must have

$$P(\delta) := \frac{1}{2}(\delta - 1)^2 + \frac{1}{4} - \left(\frac{1}{4} + \frac{\delta}{5}\right)^2 \leq 0.$$

However, the last inequality never holds for real  $\delta$  as the discriminant of the polynomial  $P$  is negative. The contradiction derived shows that the assumption  $\alpha_0 < 0$  was wrong, and the theorem is proved.  $\square$

Before proceeding with the proof of Theorem 2.3, we refine the statements of Lemmata 4.3 and 4.1 for the case where  $\Lambda \in \mathfrak{L}_1$ .

**Lemma 5.2.** *Assume that  $\Lambda \in \mathfrak{L}_1$ ; then the function  $\psi_\Lambda$  of (2.2) belongs to  $W_2^1(0, 2)$ .*

*Proof.* We recall that  $\psi_\Lambda$  is given by the series

$$\sum_{n=0}^{\infty} [\cos(\lambda_n x) - \cos(\pi n x)],$$

which converges in the topology of  $L_2(0, 2)$ . We write

$$\begin{aligned} \cos(\lambda_n x) - \cos(\pi n x) &= \left[ \cos(\mu_n x) - 1 + \frac{\mu_n^2 x^2}{2} \right] \cos(\pi n x) - [\sin(\mu_n x) - \mu_n x] \sin(\pi n x) \\ &\quad - \frac{\mu_n^2 x^2}{2} \cos(\pi n x) - \mu_n x \sin(\pi n x) \\ &=: \omega_{1,n}(x) - \omega_{2,n}(x) - \frac{\mu_n^2 x^2}{2} \cos(\pi n x) - \mu_n x \sin(\pi n x), \end{aligned}$$

where as usual  $\mu_n$  equals  $\lambda_n - \pi n$ .

By definition,  $\Lambda \in \mathfrak{L}_1$  means that  $\mu_n = c/n + \nu_n/n$  for some  $c \in \mathbb{R}$  and an  $\ell_2$ -sequence  $(\nu_n)$ , and this representation implies that the series  $\sum_{n \in \mathbb{Z}_+} \mu_n \sin(\pi n x)$  converges in  $L_2(0, 2)$  to a function from  $W_2^1(0, 2)$ . Indeed, we have

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n x)}{n} = \frac{\pi}{2}(1 - x), \quad x \in (0, 2),$$

the equality being understood in the  $L_2$ -sense, while the series

$$\sum_{n=1}^{\infty} \frac{\nu_n}{n} \sin(\pi n x)$$

converges in  $W_2^1(0, 2)$ . Similarly, the series  $\sum_{n=0}^{\infty} \mu_n^2 \cos(\pi n x)$  converges in  $W_2^1(0, 2)$ . Next, we observe that

$$\omega'_{1,n}(x) = O(n^{-3}), \quad \omega'_{2,n}(x) = O(n^{-2})$$

as  $n \rightarrow \infty$ , so that the series

$$\sum_{n=0}^{\infty} \omega_{1,n}(x), \quad \sum_{n=0}^{\infty} \omega_{2,n}(x)$$

converge in  $C^1[0, 2]$ . Summing up, we see that the function

$$\psi_\Lambda = \sum_{n=0}^{\infty} \omega_{1,n} - \sum_{n=0}^{\infty} \omega_{2,n} - \frac{x^2}{2} \sum_{n=0}^{\infty} \mu_n^2 \cos(\pi n x) - x \sum_{n=0}^{\infty} \mu_n \sin(\pi n x)$$

belongs to  $W_2^1(0, 2)$ , and the lemma is proved.  $\square$



**Lemma 5.3.** *Assume that  $\Lambda \in \mathfrak{L}_1$  and that a sequence  $(c_n)_{n \in \mathbb{Z}_+}$  from  $\ell_2$  is such that the series*

$$(5.4) \quad \sum_{n \geq 0} c_n \cos(\lambda_n x)$$

*converges in  $L_2(0, 1)$  to a function from  $W_2^1(0, 1)$ . Then this series converges in  $L_2(0, 2)$  to a function from  $W_2^1(0, 2)$ .*

*Proof.* Arguing as in the proof of Lemma 5.2, we can show that under the assumptions of the lemma the series

$$(5.5) \quad \sum_{n \geq 0} c_n [\cos(\lambda_n x) - \cos \pi n x]$$

converges in  $L_2(a, a+1)$  to a function from  $W_2^1(a, a+1)$  for any real  $a$ . In particular,  $c_n$  are cosine Fourier coefficients of a  $W_2^1(0, 1)$ -function and thus by integration by parts are easily shown to have the form

$$c_n = \frac{c}{n} + \frac{\nu_n}{n}$$

with  $c \in \mathbb{R}$  and a sequence  $(\nu_n)$  falling into  $\ell_2$ . This implies that the series

$$\sum_{n \geq 0} c_n \cos(\pi n x)$$

converges in  $L_2(a, a+1)$  to a function from  $W_2^1(a, a+1)$  for any real  $a$  (cf. the proof of Lemma 5.2). Since the same statement holds for series (5.5), the required convergence result for series (5.4) on the interval  $(0, 2)$  follows.  $\square$

*Proof of Theorem 2.3.* We have to prove that in the case where  $\sigma_0 \in W_2^1(0, \frac{1}{2})$ ,  $\Lambda \in \mathfrak{L}_1$ , and  $\phi_0 \in \Pi_\Lambda$  the function  $\sigma$  constructed in Theorem 2.1 and extending  $\sigma_0$  belongs to  $W_2^1(0, 1)$ . In view of Theorem 3.4 this is equivalent to showing that the function  $\phi$  of (3.3) belongs to  $W_2^1(0, 2)$ .

Observe first that the function  $\phi_0$  of (2.3) is the restriction of  $\phi$  to the interval  $(0, 1)$  and that  $\phi_0$  belongs to  $W_2^1(0, 1)$ . To prove this, we extend  $\sigma_0$  to some function  $\tilde{\sigma}$  from  $W_2^1(0, 1)$  and construct the function  $\tilde{\phi}$  corresponding to  $\tilde{\sigma}$  according to (3.3). By Theorem 3.4  $\tilde{\phi}$  is in  $W_2^1(0, 2)$ , and it remains to notice that, in view of Remark 3.5,  $\phi_0$  is the restriction of  $\tilde{\phi}$  onto  $(0, 1)$ , so that  $\phi_0 \in W_2^1(0, 1)$ .

In virtue of the results of Section 3 the function  $\phi$  of (3.3) has the form (3.5), i.e.,

$$\phi(s) = \sum_{n=0}^{\infty} [\alpha_n \cos(\lambda_n s) - \cos(\pi n s)] + \frac{1}{2}.$$

The restriction  $\phi_0$  of  $\phi$  to  $(0, 1)$  is therefore given in  $L_2(0, 1)$  by the same series. The assumption  $\phi_0 \in \Pi_\Lambda$  now implies that  $\alpha_n = 1 + \beta_n > 0$  with  $(\beta_n) \in \ell_2$ , so that  $\phi$  can be represented as

$$\begin{aligned} \phi(s) &= \sum_{n=0}^{\infty} [\cos(\lambda_n s) - \cos(\pi n s)] + \sum_{n=0}^{\infty} \beta_n \cos(\lambda_n s) + \frac{1}{2} \\ &=: \psi_\Lambda(s) + \psi_1(s) + \frac{1}{2}. \end{aligned}$$

Since  $\Lambda \in \mathfrak{L}_1$ , by virtue of Lemma 5.2 the function  $\psi_\Lambda$  belongs to  $W_2^1(0, 2)$ . Restricting the above equality to  $(0, 1)$  and recalling that  $\phi_0 \in W_2^1(0, 1)$ , we see that  $\psi_1 \in W_2^1(0, 1)$ . Applying now Lemma 5.3 to the series  $\sum_{n \geq 0} \beta_n \cos(\lambda_n s)$ , we conclude

that this series converges in  $L_2(0, 2)$  to a function from  $W_2^1(0, 2)$ . Thus  $\psi_1 \in W_2^1(0, 2)$ , and hence  $\phi \in W_2^1(0, 2)$  as required. The theorem is proved.  $\square$

#### APPENDIX A. PROOF OF THEOREM 3.4

We denote by  $G_2$  the subspace of all functions  $k \in L_2((0, 1) \times (0, 1))$ , for which the mappings

$$(A.1) \quad x \mapsto k(x, \cdot) \in L_2(0, 1), \quad t \mapsto k(\cdot, t) \in L_2(0, 1)$$

belong to  $C([0, 1], L_2(0, 1))$  (i.e., for which these mappings coincide a.e. with continuous ones). We also denote by  $G_2^1$  the subspace of  $G_2$  consisting of those  $k$ , for which mappings (A.1) belong to  $C^1([0, 1], L_2(0, 1))$ . The spaces of integral operators with kernels from  $G_2$  (respectively, from  $G_2^1$ ) will be denoted by  $\mathfrak{G}_2$  (respectively, by  $\mathfrak{G}_2^1$ ).

**Lemma A.1.** *Assume that  $\phi \in W_2^1(0, 2)$ ; then the operator  $F_\phi$  with kernel  $f_\phi(x, t) := \phi(x+t) + \phi(|x-t|)$  belongs to  $\mathfrak{G}_2^1$ .*

*Proof.* Since the function  $k(x, t) := \phi'(x+t)$  obviously belongs to  $G_2$ , we use the equalities

$$\phi(x+t) = \phi(t) + \int_0^x \phi'(t+\xi) d\xi = \phi(x) + \int_0^t \phi'(x+\xi) d\xi$$

to justify the inclusion  $\phi(x+t) \in G_2^1$ . In the same manner we show that  $\phi(|x-t|) \in G_2^1$ , which by definition yields  $F_\phi \in \mathfrak{G}_2^1$ .  $\square$

**Lemma A.2.** *Assume that  $R \in \mathfrak{G}_2^1$  and that the operator  $I+R$  is invertible. Then the integral operator  $\hat{R} := (I+R)^{-1} - I$  belongs to  $\mathfrak{G}_2^1$ .*

*Proof.* We use a kind of the bootstrap method based on the formula

$$(A.2) \quad \hat{R} = R\hat{R}R + R^2 - R,$$

which follows from the relation  $\hat{R}R + \hat{R} + R = 0$ . Namely, using (A.2), we first show that  $\hat{R}$  is a Hilbert–Schmidt operator, then that  $\hat{R} \in \mathfrak{G}_2$ , and finally that  $\hat{R} \in \mathfrak{G}_2^1$ .

Since  $R \in \mathfrak{G}_2^1 \subset \mathfrak{G}_2$  and  $\mathfrak{G}_2$  is an ideal in the algebra of all bounded operators, we conclude from (A.2) that  $\hat{R} \in \mathfrak{G}_2$ .

Next we show that  $\hat{R}$  belongs to  $\mathfrak{G}_2^1$ . To this end it suffices to prove the inclusions  $\mathfrak{G}_2 \cdot \mathfrak{G}_2 \cdot \mathfrak{G}_2 \subset \mathfrak{G}_2$  and  $\mathfrak{G}_2 \cdot \mathfrak{G}_2^1 \subset \mathfrak{G}_2^1$ . Assume that  $R_1, R_3 \in \mathfrak{G}_2$ ,  $R_2 \in \mathfrak{G}_2$ , and let  $R_{ij} := R_i R_j$  and  $R_{123} = R_1 R_2 R_3$ . Then the kernel  $r_{12}$  of the operator  $R_{12}$  is given by

$$r_{12}(x, t) = \int_0^1 r_1(x, s) r_2(s, t) ds = (r_1(x, \cdot), \overline{r_2(\cdot, t)})_{L_2(0,1)},$$

and, using the Cauchy–Schwarz–Bunyakovski inequality, we find that  $r_{12}(x, \cdot)$  belongs to  $L_2(0, 1)$  and

$$\|r_{12}(x, \cdot)\|_{L_2(0,1)} \leq \|r_1(x, \cdot)\|_{L_2(0,1)} \|r_2\|_{L_2((0,1)^2)}.$$

By linearity we also get

$$\|r_{12}(x, \cdot) - r_{12}(x', \cdot)\|_{L_2(0,1)} \leq \|r_1(x, \cdot) - r_1(x', \cdot)\|_{L_2(0,1)} \|r_2\|_{L_2((0,1)^2)},$$

which yields continuity of the mapping  $[0, 1] \ni x \mapsto r_{12}(x, \cdot) \in L_2(0, 1)$ . Similar arguments show that the kernels  $r_{13}$  and  $r_{23}$  belong to  $G_2$  (in fact, they are even continuous on  $(0, 1)^2$ ). Relation A.2 now implies that  $\hat{R} \in \mathfrak{G}_2^1$ .

Assume now that  $R_1, R_3 \in \mathfrak{G}_2^1$  and  $R_2 \in \mathfrak{G}_2$ ; then

$$\frac{dr_{12}(x, t)}{dx} = \frac{d}{dx} (r_1(x, \cdot), \overline{r_2(\cdot, t)})_{L_2(0,1)} = (r_1'(x, \cdot), \overline{r_2(\cdot, t)})_{L_2(0,1)}$$

and thus  $r_{12}$  is continuous in  $t$  and once continuously differentiable in  $x$ . Similar arguments show that  $R_1 R_2 R_3 \in \mathfrak{G}_2^1$  and  $R_1 R_3 \in \mathfrak{G}_2^1$ . Using this observation in (A.2), we conclude that  $\hat{R} \in \mathfrak{G}_2^1$ , and the lemma is proved.  $\square$

**Lemma A.3.** *Assume that  $\phi \in \Phi \cap W_2^1(0, 2)$  and*

$$(A.3) \quad I + F_\phi = (I + K)^{-1}(I + K^\top)^{-1}, \quad K \in \mathfrak{G}_2.$$

*Then the kernels  $k$  and  $l$  of the operators  $K$  and  $L := (I + K)^{-1} - I$  respectively have the following property: for  $0 \leq t \leq x \leq 1$ ,*

$$(A.4) \quad \begin{aligned} k(x, t) &= -f_\phi(x, t) + k_1(x, t), \\ l(x, t) &= f_\phi(x, t) + l_1(x, t), \end{aligned}$$

*where the kernels  $k_1$  and  $l_1$  are continuously differentiable.*

*Proof.* Applying  $I + K$  to both sides of equation (A.3), rewriting the resulting equality in terms of kernels, and recalling that  $K^\top$  has an upper-diagonal kernel, we arrive at the so-called Gelfand–Levitan–Marchenko (GLM) equation

$$k(x, t) + f_\phi(x, t) + \int_0^x k(x, s)f_\phi(s, t) ds = 0, \quad 0 \leq t \leq x \leq 1.$$

Fixing  $x \in (0, 1)$  in the GLM equation and denoting

$$g_x(t) := \begin{cases} f_\phi(x, t) & \text{if } 0 < t < x < 1, \\ 0 & \text{if } 0 < x \leq t < 1, \end{cases}$$

we conclude that

$$(I + F_\phi)k(x, \cdot) = -g_x.$$

Since the operator  $I + F_\phi$  is invertible and  $F_\phi \in \mathfrak{G}_2^1$  by virtue of Lemma A.1, Lemma A.2 implies that the operator  $R := (I + F_\phi)^{-1} - I$  belongs to  $\mathfrak{G}_2^1$ . In particular, with  $r$  being the kernel of  $R$ , we have for  $x > t$

$$k(x, t) = -g_x(t) - \int_0^1 r(t, s)g_x(s) ds = -f_\phi(x, t) - \int_0^x r(t, s)f_\phi(x, s) ds.$$

Since the function

$$k_1(x, t) := - \int_0^x r(t, s)f_\phi(x, s) ds$$

is easily seen to be continuously differentiable on  $(0, 1)^2$ , the required representation follows.

The operator  $L$  satisfies the relation

$$L = F_\phi + K^\top + F_\phi K^\top,$$

or, in terms of kernels,

$$l(x, t) = f_\phi(x, t) + \int_0^t f_\phi(x, s)k(t, s) ds, \quad 0 \leq t \leq x \leq 1.$$

Now the derived representation (A.4) for the kernel  $k$  implies that the function

$$l_1(x, t) := \int_0^t f_\phi(x, s)k(t, s) ds$$

is continuously differentiable on  $(0, 1)^2$ , and the proof is complete.  $\square$

*Proof of Theorem 3.4.* Assume that  $\sigma \in W_2^1(0, 1)$  and show that then the function  $\phi$  of (3.3) belongs to  $W_2^1(0, 2)$ . To this end it suffices to prove that the kernel  $l_\sigma$  has suitable smoothness properties.

We recall that the function  $a(u, v) := l_\sigma(\frac{u+v}{2}, \frac{u-v}{2})$  is continuous on the set  $\tilde{\Omega}^+ := \{(u, v) \mid 0 \leq v \leq u \leq 2\}$  and satisfies there the integral equation

$$\begin{aligned} a(u, v) = & -\frac{1}{4} \int_v^u d\alpha \int_0^v q\left(\frac{\alpha - \beta}{2}\right) a(\alpha, \beta) d\beta \\ & - \frac{1}{2} \int_0^v d\alpha \int_0^\alpha q\left(\frac{\alpha - \beta}{2}\right) a(\alpha, \beta) d\beta \\ & - \frac{1}{2} \left[ \int_0^{u/2} q(\alpha) d\alpha + \int_0^{v/2} q(\beta) d\beta \right] \end{aligned}$$

with  $q := \sigma'$ . This integral equation implies that the function

$$\tilde{a}(u, v) := a(u, v) + \frac{1}{2} \left[ \int_0^{u/2} q(\alpha) d\alpha + \int_0^{v/2} q(\beta) d\beta \right]$$

is continuously differentiable in  $\tilde{\Omega}^+$ . As a result,  $l_\sigma$  has the representation

$$l_\sigma(x, t) = \tilde{a}(x + t, x - t) - \frac{1}{2}\sigma\left(\frac{x+t}{2}\right) - \frac{1}{2}\sigma\left(\frac{x-t}{2}\right) + \sigma(0),$$

which implies that the function  $\int_0^x l_\sigma^2(x, t) dt$  is in  $W_2^1(0, 1)$ , so that  $\phi \in W_2^1(0, 1)$  as well.

Conversely, let  $\phi \in \Phi \cap W_2^1(0, 2)$  and determine  $\sigma$  through relation (3.3). In other words, with an operator  $K \in \mathfrak{S}_2^+$  satisfying (A.3) and  $l$  being the kernel of the operator  $L := (I + K)^{-1} - I \in \mathfrak{S}_2^+$ , we have

$$\sigma(x) = -2\phi(2x) + 2 \int_0^x l^2(x, t) dt, \quad x \in (0, 1).$$

Using representation (A.4), one can easily show that the function  $\int_0^x l^2(x, t) dt$  belongs to  $W_2^1[0, 1]$ . Henceforth  $\sigma \in W_2^1(0, 1)$ , and the proof is complete.  $\square$

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